A Study in the Integral of Sine and Cosine Functions

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Abstract
Trigonometric functions are among the most widely used functions in many science fields, especially sine and cosine functions because they are essential for periodic functions that describe sound and light waves in different types and wavelengths. Therefore, researchers studied the integrals of sine and cosine functions in different forms of the integrating function. In this paper, we spotlighted several most important yet under-studied integrals that are poorly mentioned in Arabic and foreign textbooks and studies. In addition, we studied Integral of Sine and Cosine for **n** as a positive rational number and concluded that each of these integrals leads to functional series. When studying the convergence of these series using the D'Alembert ratio test, we found that these series are convergent over the entire set of real numbers. This convergence is highly useful when applying such integrals in different science fields.

Keywords: Power Series; Sine; Cosine; Convergence; Integral; D'Alembert Ratio Test.

1. Introduction

The computation of integration has emerged because of the need to find a general way to assign spaces, sizes and heavyweights. This method was used in its initial form by Archimedes, and developed regularly in the 17th century by Cavallery, Torricelli, Verma, Pascal and other scientists, and in 1659 Barrow created the relationship between the delimitation of the area and the designation of the tangent, after which Newton and Lipentz in the 1770s separated this relationship from previous engineering issues. Therefore, the relationship between the integration and calculus was found, a Newton and Lipentz and their students used this relationship to develop the methods of the integration process. In addition to Euler's -work, which has a great credit for the integration calculation methods getting to where they are now [1] In view of the great importance of the sine and the cosine in all domains of life, being essential in the study of periodic functions that describe sound and light waves of various types and wavelengths. New formulas had to be identified for both the sine and cosine, with a view to applying each of these formulas in a different domain.

We mention some of the familiar formulas for both sine and cosine as shown in references [1-15]:

$$\int \sin\frac{x}{2}\,dx,\ \int\frac{dx}{\sin^2 x},\ \int\frac{dx}{\sin x},\ ....$$

$$\int \cos\frac{x}{2}\,dx,\ \int\frac{dx}{\cos^2 x},\ \int\frac{dx}{\cos x},\ ....$$

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In a very limited number of references, a study that defines the value of both integrations $\int_{0}^{+\infty} \sin x^2 dx$ $\int_{0}^{+\infty} \cos x^2 dx$ ،was presented, based on the integration of Euler--Boisson, and other references in which examples calculate the specific calculus of the sine and the cosine of the power series([16], [17], [18]).

Research Objective:

To shed a light on a range of integrations used during the calculation of integrations $\int_{0}^{+\infty} \cos x^2 dx$ $\int_{0}^{+\infty} \sin x^2 dx$ using the Euler--Boisson method, as well as to explain the way in which value was obtained for integrators. Through examples, the method of calculating integrations $\int_{b}^{a} \sin x^2 dx$ $\int_{a}^{b} \cos x^2 dx$ and has been clarified. [16], [17] [18]

To generalize the previous study, we studied the integration of the two sine and cosine functions in the case where the integration is indefinite and the angle x is raised to exponent n therefore $\int \cos x^n dx$ $\int \sin x^n dx$, so that n is a positive relative number and we found that each of them is a power series ,When studying the convergence of these sequences using D'Alembert Ratio Test , we found that they were converging on the whole set of real numbers. and this convergence is very useful when using such integrations in applied fields.

2. Discussion

We begin by showing what previous studies of the two sine and cosine integrations have found.

1-Calculation of integrations $\int_{0}^{+\infty} \cos x^2 dx$ and $\int_{0}^{+\infty} \sin x^2 dx$ from the following integrations: [16], [17]

Euler- -Boisson Integration

1) $\int_{0}^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

2) $\int_{0}^{+\infty} e^{-xy} \sin x dx = \frac{1}{1+y^2}$ $\quad y > 0$

3) $\int_{0}^{+\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{\pi}}$

View the method for

$$I = \int_{0}^{+\infty} \sin x^2 dx$$

We make a change in the variant, we impose:

y = x2⇒ x = $\sqrt{y}$ ⇒ dx = $\frac{dy}{2\sqrt{y}}$

Substitute for integration:

$$I = \frac{1}{2}\int_{0}^{+\infty} \frac{\sin y}{\sqrt{y}} dy \qquad (*)$$

We multiply the second part of the previous relationship with:

$$\lim_{k\to 0} e^{-ky} = 1$$

We get:

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I = $\frac{1}{2}\int_0^{+\infty} e^{-ky}\frac{\sin y}{\sqrt{y}}dy$

From the integration of Euler- Boisson we get:

$$\frac{1}{\sqrt{y}} = \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-x^2}dx$$

Substitute for integration:

I = $\frac{1}{\sqrt{\pi}}\lim_{k\to 0}\int_0^{+\infty} dy\int_0^{+\infty} e^{-y(k+x^2)}\sin y dy$

We rearrange integrations and replace:

$$\int_0^{+\infty} e^{-xy}\sin x dx = \frac{1}{1+y^2} \quad ; \quad y > 0$$

We get:

I = $\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{dx}{1+x^4}$

But:

$$\int_0^{+\infty}\frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{\pi}}$$

$\Rightarrow$I = $\int_0^{+\infty}\sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

In the same way we get:

$$\int_0^{+\infty}\cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

2- Calculation of $\int_a^b \cos x^2 dx$ and $\int_a^b \sin x^2 dx$ Integrations: [18]

We explain the method through the following example: calculate $\int_0^1 \cos x^2 dx$ value for accuracy 0.001.

Based on the sine function in the power series we reach the desired. [4] ‹[5] ‹[11] ‹[12]

We have:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

It is a series of converging forces over the group of real numbers, a radius approaching R = ∞, and thus it is:

$$\cos x^2 = 1 - + \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots + (-1)^n\frac{x^{4n}}{(2n)!} + \ldots$$

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$$\int_0^1 \cos x^2 dx = \int_0^1 \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots + (-1)^n \frac{x^{4n}}{(2n)!} + \ldots\right) dx$$

$$\int_0^1 \cos x^2 dx = \left(x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{13\cdot 6!} + \ldots\right)\Bigg|_0^1$$

We neglect the limit $\frac{x^{13}}{13\cdot 6!}$ and the limits that follow, we get:

$$\int_0^1 \cos \mathrm{x}^2 \mathrm{dx} = 1 - \frac{1}{10} + \frac{1}{216} \approx 0.905$$

In the same way, we calculate $\int_b^a \sin x^2 dx$ for any values of a and b.

Using the sine series in the following power series : [4], [5], [11], [12]:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + - - - - -$$

In this research, we studied the unspecified integration of the two sine and cosine function when the angle of x is up to n, where n is a positive relative number.

First: Sine Function $\int \sin x^n dx$

1. Integration calculation $\int \sin x^2 dx$ (n = 2 even number) ,

To calculate integration $I_2 = \int \sin x^2 dx$ We make a change in the variable:

$$\mathrm{X}^2 = \mathrm{t} \Rightarrow \mathrm{x} = \sqrt{t} \Rightarrow \mathrm{dx} = \frac{dt}{2\sqrt{t}}$$

Substitute for integration:

$$I_2 = \int \sin x^2 dx = \frac{1}{2}\int \frac{1}{\sqrt{t}} \sin t\ dt = \frac{1}{2}\int t^{-\frac{1}{2}} \sin t\ dt$$

Using the sin t sine series in the power chain:

$$\sin x \approx \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \qquad (1)$$

Which is a close sequence on the set of real numbers and a radius that is close to R = ∞.

Substitute for integration:

$$I_2 = \frac{1}{2}\int t^{-\frac{1}{2}}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right] dt = \frac{1}{2}\left[\frac{2t^{\frac{3}{2}}}{3} - \frac{2t^{\frac{73}{2}}}{7\cdot 3!} + \frac{2t^{\frac{11}{2}}}{11\cdot 5!} + \ldots\right]$$

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Thus:

$$I_2 = \frac{1}{3}t^{\frac{3}{2}} - \frac{1}{7\cdot 3!}t^{\frac{7}{2}} + \frac{1}{11\cdot 5!}t^{\frac{11}{2}} - \frac{1}{15\cdot 7!}t^{\frac{15}{2}}$$

We substitute $t = x^2$ we get:

$$\int \sin x^2 dx \approx \left(\frac{x^3}{3} - \frac{x^7}{3!\cdot 7} + \frac{x^{11}}{5!\cdot 11} + \frac{x^{15}}{7!\cdot 15} + \ldots\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)}$$

2. Integration calculation $\int \sin x^3 dx$ (n = 3 odd number),

To calculate integration $I_3 = \int \sin x^3 dx$ We making a change in variable, we impose:

$$x^3 = t \Rightarrow x = \sqrt[3]{t} = t^{\frac{1}{3}}$$

$$3x^2 dx = dt \Rightarrow dx = \frac{dt}{3x^2} = \frac{dt}{3t^{\frac{2}{3}}} = \frac{1}{3}t^{\frac{-2}{3}} dt$$

Substitute for Integration

$$I_3 = \int \sin x^3 dx = \frac{1}{3}\int \sin t \frac{dt}{\sqrt[3]{t}} = \frac{1}{3}\int t^{\frac{-2}{3}} \sin t \, dt$$

Benefiting relationship (1) we get:

$$I_3 = \frac{1}{3}\int t^{\frac{-2}{3}} \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + - - - - - \right)dt$$

$$I_3 = \frac{1}{3}\int \left(t^{\frac{1}{3}} - \frac{1}{3!}t^{\frac{7}{3}} + \frac{1}{5!}t^{\frac{13}{3}} - \frac{1}{7!}t^{\frac{19}{3}} + - - - \right)dt$$

$$I_3 = \frac{1}{3}\left(\frac{3}{4}t^{\frac{4}{3}} - \frac{3}{3!\cdot 10}t^{\frac{10}{3}} + \frac{3}{5!\cdot 16}t^{\frac{16}{3}} - \frac{3}{7!\cdot 22}t^{\frac{22}{3}} + - - - - \right)$$

$$I_3 = \frac{1}{4}t^{\frac{4}{3}} - \frac{1}{3!\cdot 10}t^{\frac{10}{3}} + \frac{1}{5!\cdot 16}t^{\frac{16}{3}} - \frac{1}{7!\cdot 22}t^{\frac{22}{3}} + - - - -$$

We substitute $t = x^3$ we get:

$$\int \sin x^3 = \frac{1}{4}(x^3)^{\frac{4}{3}} - \frac{1}{3!\cdot 10}(x^3)^{\frac{10}{3}} + \frac{1}{5!\cdot 16}(x^3)^{\frac{16}{3}} - \frac{1}{7!\cdot 22}(x^3)^{\frac{22}{3}} + - - - -$$

$$\int \sin x^3 = \frac{1}{4}x^4 - \frac{1}{3!\cdot 10}x^{10} + \frac{1}{5!\cdot 16}x^{16} - \frac{1}{7!\cdot 22}x^{22} + - - - -$$

$$\int \sin x^3 \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(6n+4)} x^{(6n+4)}$$

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3. Integration $\int \sin x^n dx$,To calculate integration $I_n = \int \sin x^n dx$ We impose:

$$x^n = t \Rightarrow x = \sqrt[n]{t} = t^{\frac{1}{n}} \Rightarrow dx = \frac{1}{n} t^{\frac{1-n}{n}} dt$$

$$\Rightarrow I_n = \int \sin x^n dx = \frac{1}{n} \int t^{\frac{1-n}{n}} \sin t dt$$

Benefiting relationship (1) we get:

$$I_n = \frac{1}{n} \int \left(t^{\frac{1}{n}} - \frac{1}{3!} t^{\frac{2n+1}{n}} + \frac{1}{5!} t^{\frac{4n+1}{n}} - \frac{1}{7!} t^{\frac{6n+1}{n}} + \dots \right) dt$$

By calculating integration and compensation for t and its equal we get:

$$I_n = \frac{1}{1!} \cdot \frac{1}{n+1} x^{n+1} - \frac{1}{3!} \frac{1}{3n+1} x^{3n+1} + \frac{1}{5!} \cdot \frac{1}{5n+1} x^{5n+1} - \frac{1}{7!} \cdot \frac{1}{7n+1} x^{7n+1} + \dots$$

$$\Rightarrow \int \sin x^n dx \approx \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)! [n(2k+1)+1]} x^{n(2k+1)+1}$$

It is a power series that its limits are known functions on the field (-∞, +∞), we study convergence using the D'Alembert test. [1], [4], [5], [12] We take:

$$\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = q$$

We distinguish the following cases:

a. If: q < 1 sequence is close.
b. q > 1 sequence is spaced.
c. q = 1 It is possible to be spaced or close.

$$\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{(2nk+n+1)\left|x^{2n}\right|}{(2k+3)(2k+2)(2nk+3n+1)}$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = 0 < 1$$

Thus, the sequence converges on the field (-∞, +∞).

Second: The Function of Cosine $\int \cos x^n dx$

1. Integration calculation (n = 2 even numbers) ,Integration calculation $J_2 = \int \cos x^2 dx$

We make a change in the variable.We impose

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$$x^2 = t \Rightarrow x = \sqrt{t} = t^{\frac{1}{2}}$$

$$2xdx = dt \Rightarrow dx = \frac{dt}{2\sqrt{t}} = \frac{1}{2} t^{\frac{-1}{2}} dt$$

Substitute for integration

$$J_2 = \int \cos x^2 dx = \frac{1}{2}\int \cos t \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{\frac{-1}{2}} \cos t \, dt$$

Using the function cost in the power series:

$$\cos t \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \qquad (2)$$

It is a close sequence on the set of real numbers and a radius that is close to R = ∞.

Substitute for integration:

$$J_2 = \frac{1}{2}\int (t^{\frac{-1}{2}} - \frac{1}{2!} t^{\frac{3}{2}} + \frac{1}{4!} t^{\frac{7}{2}} - \frac{1}{6!} t^{\frac{11}{2}} + ----)dt$$

$$J_2 = \frac{1}{2}(\frac{2}{1} t^{\frac{1}{2}} - \frac{1}{2!}\cdot\frac{2}{5} t^{\frac{5}{2}} + \frac{1}{4!}\cdot\frac{2}{9} t^{\frac{9}{2}} - \frac{1}{6!}\cdot\frac{2}{13} t^{\frac{13}{2}} + ------)$$

$$J_2 = (t^{\frac{1}{2}} - \frac{1}{2!}\cdot\frac{1}{5} t^{\frac{5}{2}} + \frac{1}{4!}\cdot\frac{1}{9} t^{\frac{9}{2}} - \frac{1}{6!}\cdot\frac{1}{13} t^{\frac{13}{2}} + ------)$$

By substituting $t = x^2$ we get:

$$J_2 = ((x^2)^{\frac{1}{2}} - \frac{1}{2!}\cdot\frac{1}{5}(x^2)^{\frac{5}{2}} + \frac{1}{4!}\cdot\frac{1}{9}(x^2)^{\frac{9}{2}} - \frac{1}{6!}\cdot\frac{1}{13}(x^2)^{\frac{13}{2}} + ------)$$

$$J_2 = (x - \frac{1}{2!}\cdot\frac{x^5}{5} + \frac{1}{4!}\cdot\frac{x^9}{9} - \frac{1}{6!}\cdot\frac{x^{13}}{13} + -----)$$

$$\int \cos x^2 dx \approx x + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!\,(4k+1)} x^{(4k+1)}$$

2. Integration $J_3 = \int \cos x^3 dx$ calculation (n = 3 odd number)

To calculate integration $J_3 = \int \cos x^3 dx$

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We make a change in the variable, we impose:

$$x^3 = t \Rightarrow x = \sqrt[3]{t} = t^{\frac{1}{3}}$$

$$3x^2 dx = dt \Rightarrow dx = \frac{dt}{3x^2} = \frac{dt}{3t^{\frac{2}{3}}} = \frac{1}{3} t^{\frac{-2}{3}} dt$$

Substitute for integration

$$J_3 = \int \cos x^3 dx = \frac{1}{3}\int \cos t \frac{dt}{\sqrt[3]{t}} = \frac{1}{3}\int t^{\frac{-2}{3}} \cos t \, dt$$

Using relationship (2)

$$J_3 = \frac{1}{3}\int t^{\frac{-2}{3}} (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + ----)dt$$

$$J_3 = \frac{1}{3}\int (t^{\frac{-2}{3}} - \frac{1}{2!} t^{\frac{4}{3}} + \frac{1}{4!} t^{\frac{10}{3}} - \frac{1}{6!} t^{\frac{16}{3}} + -----)dt$$

$$J_3 = \frac{1}{3}(\frac{3}{1} t^{\frac{1}{3}} - \frac{1}{2!}\cdot\frac{3}{7} t^{\frac{7}{3}} + \frac{1}{4!}\cdot\frac{3}{13} t^{\frac{13}{3}} - \frac{1}{6!}\cdot\frac{3}{19} t^{\frac{19}{3}} + -----)$$

$$J_3 = (t^{\frac{1}{3}} - \frac{1}{2!}\cdot\frac{1}{7} t^{\frac{7}{3}} + \frac{1}{4!}\cdot\frac{1}{13} t^{\frac{13}{3}} - \frac{1}{6!}\cdot\frac{1}{19} t^{\frac{19}{3}} + -----)$$

By substituting $t = x^3$ we get:

$$J_3 = ((x^3)^{\frac{1}{3}} - \frac{1}{2!}\cdot\frac{1}{7}(x^3)^{\frac{7}{3}} + \frac{1}{4!}\cdot\frac{1}{13}(x^3)^{\frac{13}{3}} - \frac{1}{6!}\cdot\frac{1}{19}(x^3)^{\frac{19}{3}} + -----)$$

$$J_3 = (x - \frac{1}{2!}\cdot\frac{x^7}{7} + \frac{1}{4!}\cdot\frac{x^{13}}{13} - \frac{1}{6!}\cdot\frac{x^{19}}{19} + -----)$$

$$\int \cos x^3 dx \approx x + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!\,(6k+1)} x^{(6k+1)}$$

3. Integration $\int \cos x^n dx$.

To calculate integration $J_n = \int \cos x^n dx$ We make a change in the variable, we impose:

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$$x^n = t \Rightarrow x = \sqrt[n]{t} = t^{\frac{1}{n}}$$

$$dt = nx^{(n-1)}dx \Rightarrow dx = \frac{1}{n} t^{(\frac{1-n}{n})} dt$$

Substitute for integration

$$J_n = \int \cos x^n dx = \frac{1}{n}\int \cos t \frac{dt}{t^{(\frac{1-n}{n})}} = \frac{1}{n}\int t^{(\frac{1-n}{n})} \cos t \, dt$$

Using Relationship (2)

$$J_n = \frac{1}{n}\int t^{(\frac{1-n}{n})}(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + ----)dt$$

$$J_n = \frac{1}{n}\int \ (t^{(\frac{1-n}{n})} - \frac{1}{2!}t^{(\frac{1-n}{n}+2)} + \frac{1}{4!}t^{(\frac{1-n}{n}+4)} - \frac{1}{6!}t^{(\frac{1-n}{n}+6)} + ----)dt$$

$$J_n = \frac{1}{n}\left[\int t^{\frac{1-n}{n}} dt - \frac{1}{2!}\int t^{\frac{n+1}{n}} dt + \frac{1}{4!}\int t^{\frac{3n+1}{n}} dt - \frac{1}{6!}\int t^{\frac{5n+1}{n}} dt + ---\right]$$

$$J_n = t^{\frac{1}{n}} - \frac{1}{2!}\frac{1}{2n+1}t^{\frac{2n+1}{n}} + \frac{1}{4!}\cdot\frac{1}{4n+1}t^{\frac{4n+1}{n}} - \frac{1}{6!}\cdot\frac{1}{6n+1}t^{\frac{6n+1}{n}} + --$$

By substituting $t = x^n$ we get:

$$J_n = t^{\frac{1}{n}} - \frac{1}{2!}\frac{1}{2n+1}x^{2n+1} + \frac{1}{4!}\cdot\frac{1}{4n+1}x^{4n+1} - \frac{1}{6!}\cdot\frac{1}{6n+1}x^{6n+1} + --$$

$$J_n \approx x + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!(2nk+1)}x^{2nk+1}$$

It is a power series that its limits are known functions on the field (-∞, +∞), we study convergence using the D'Alembert test. [1], [4], [5], [12] We take:

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = q$$

We Distinguish the Following Cases:

a. If: q < 1 sequence is close.
b. q > 1 sequence is spaced.

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c. $q = 1$ It is possible to be spaced or close.

We came up with a functional sequence by studying its convergence according to the D'Alembert test we find that:

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(2k)!(2nk+1)x^{2nk+2n+1}}{(2k+2)!(2nk+2n+1)x^{2nk+1}}\right| =$$

$$\lim_{k\to\infty}\frac{(2nk+1)\left|x^{2n}\right|}{(2k+2)(2k+1)(2nk+2n+1)} = 0 < 1$$

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = 0 < 1$$

Thus, the sequence is close everywhere in R.

3. Conclusions and Recommendations:

a. In this research, we found mathematical formulas for the sine and the cosine functions, when the x angle is raised to the exponent n, where n is a positive relative number, using the of each of the two functions in the power chain, where these formulas can be used in the applied fields you use.

b. From the study in the research, we note that we can benefit from the change in the studied formula $\lim_{k\to 0} e^{-kx} = 1$ to become a new formula that enables us to reach the desired.

c. Calculated values for the following integrations can be used:

$$\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{+\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

$$\int_0^{+\infty} \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_0^{+\infty} \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

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